

THE STRONG ELLIPTIC MAXIMUM PRINCIPLE FOR VECTOR BUNDLES AND APPLICATIONS TO MINIMAL MAPS

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ABSTRACT. Based on works by Hopf, Weinberger, Hamilton and Evans, we state and prove the strong elliptic maximum principle for smooth sections in vector bundles over Riemannian manifolds and give some applications in Differential Geometry. Moreover, we use this maximum principle to obtain various rigidity theorems and Bernstein type theorems in higher codimension for minimal maps between Riemannian manifolds.

1. INTRODUCTION

The maximum principle is one of the most powerful tools used in the theory of PDEs and Geometric Analysis. In general, maximum principles for solutions of second order elliptic differential equations, that are defined in the closure of a bounded domain of the euclidean space, appear in two forms. The *weak maximum principle* states that the maximum of the solution is attained at the boundary of the domain, but in principle it might occur in the interior as well. On the other hand, the *strong maximum principle* asserts that the solution achieves its maximum only at boundary points, unless it is constant. For instance, H. Hopf [Hop27] established such strong maximum principles for a wide class of general second order differential equations. For example, he proved that if a solution u of the uniformly elliptic differential

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equation

$$\mathcal{L}u = 0, \quad \mathcal{L} = \sum_{i,j=1}^m a^{ij} \partial_{ij}^2 + \sum_{j=1}^m b^j \partial_j, \quad (*)$$

attains its supremum or infimum at an interior point of its domain D of definition, then it must be constant.

Equivalently, the above strong elliptic maximum principle of Hopf can be interpreted as follows: If a solution u of $\mathcal{L}u = 0$ maps an interior point of D to the boundary of the set $K = (\inf_D u, \sup_D u)$, then u maps any point of D to the boundary of K and hence it must be constant. For the proof of this strong maximum principle Hopf used the *Hopf Lemma*, which implies that the subset $B \subset D$ consisting of points where u attains a value in ∂K is open. Since by continuity B is also closed, one has $B = D$, if D is connected and B is non-empty.

The generalization of Hopf's maximum principle to elliptic and semi-linear parabolic systems has been first considered by H. Weinberger [Wei75]. Let us recall briefly here the elliptic version of this strong maximum principle: Assume that the vector valued map

$$u : D \subset \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad u := (u_1, \dots, u_n),$$

is a solution of the differential system

$$\mathcal{L}u + \Psi(u) = 0,$$

such that $u(D)$ is contained in a closed convex set $K \subset \mathbb{R}^n$. Here \mathcal{L} is a second order uniformly elliptic differential operator of the form given in (*), D is an open domain of \mathbb{R}^m and $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz continuous map. Suppose further that for any boundary point $y_0 \in \partial K$ the vector $\Psi(y_0)$ belongs to the tangent cone of K at y_0 (for the exact definition see Section 2.1). Under various additional assumptions on the regularity of the boundary of the convex set K , Weinberger proved that, if an interior point of D is mapped via u to a boundary point of K , then every point of D is mapped to the boundary of K . Recently, L.C. Evans [Eva10] gave a proof of Weinberger's maximum principle without imposing any regularity assumption on the boundary of the convex set K .

In his seminal papers, R. Hamilton [Ham82, Ham86] derived parabolic maximum principles for sections in Riemannian vector bundles. There one compares the solution of a parabolic differential equation with a solution of an associated ODE. The weak parabolic maximum principle

of Weinberger can be seen as a special case of Hamilton's more general maximum principle in [Ham86] since Weinberger's result follows from the application of Hamilton's maximum principle in the case of a trivial bundle. Hamilton's maximum principle appears in many different forms and became an important tool in the study of geometric evolution equations (cf. [Eck04, CCG⁺08, Bre10, AH11]).

Here we state and prove the strong elliptic maximum principle for sections in Riemannian vector bundles. This maximum principle is in the most general form and contains all the previous results by Hopf, Weinberger, Evans and it also contains the elliptic version of Hamilton's parabolic maximum principle. It turns out that it is extremely powerful and we apply it to derive optimal Bernstein type results for minimal maps between Riemannian manifolds.

In order to state the elliptic version of the strong maximum principle for sections in vector bundles, we must introduce an appropriate notion of convexity for subsets of Riemannian vector bundles. In [Ham82] Hamilton gave the following definition:

Definition 1.1. (Hamilton). *Let (E, π, M) be a vector bundle over the manifold M and let K be a closed subset of E .*

- (i) *The set K is said to be fiber-convex or convex in the fiber, if for each point x of M , the set $K_x := K \cap E_x$ is a convex subset of the fiber $E_x = \pi^{-1}(x)$.*
- (ii) *The set K is said to be invariant under parallel transport, if for every smooth curve $\gamma : [0, b] \rightarrow M$ and any vector $v \in K_{\gamma(0)}$, the unique parallel section $v(t) \in E_{\gamma(t)}$, $t \in [0, b]$, along $\gamma(t)$ with $v(0) = v$, is contained in K .*
- (iii) *A fiberwise map $\Psi : E \rightarrow E$ is a map such that $\pi \circ \Psi = \pi$, where π denotes the bundle projection. We say a fiberwise map Ψ points into K (or is inward pointing), if for any $x \in M$ and any $\vartheta \in \partial K_x$ the vector $\Psi(\vartheta)$ belongs to the tangent cone $C_{\vartheta}K_x$ of K_x at ϑ .*

Next we state the strong elliptic maximum principle for sections in Riemannian vector bundles. Throughout the paper all manifolds will be smooth and connected.

Let (E, π, M) be a vector bundle of rank k over a smooth manifold M . Suppose g_E is a bundle metric on E and that ∇ is a metric connection on E . In this paper we consider uniformly elliptic operators \mathcal{L} on $\Gamma(E)$

of second order that are given locally by

$$\mathcal{L} = \sum_{i,j=1}^m a^{ij} \nabla_{e_i, e_j}^2 + \sum_{j=1}^m b^j \nabla_{e_j}, \quad (**)$$

where $a \in \Gamma(TM \otimes TM)$ is a symmetric, uniformly positive definite tensor and $b \in \Gamma(TM)$ is a smooth vector field such that

$$a = \sum_{i,j=1}^m a^{ij} e_i \otimes e_j \quad \text{and} \quad b = \sum_{j=1}^m b^j e_j$$

in a local frame field $\{e_1, \dots, e_k\}$ of TM .

Theorem A. (Strong Elliptic Maximum Principle).

Let (M, g_M) be a Riemannian manifold and (E, π, M) a vector bundle over M equipped with a Riemannian metric g_E and a metric connection ∇ . Let K be a closed fiber-convex subset of the bundle E that is invariant under parallel transport and let $\phi \in K$ be a smooth section such that

$$\mathcal{L}\phi + \Psi(\phi) = 0,$$

where here \mathcal{L} is a uniformly elliptic operator of second order of the form given in $(**)$ and Ψ is a smooth fiberwise map that points into K . If there exists a point x_0 in the interior of M such that $\phi(x_0) \in \partial K_{x_0}$, then $\phi(x) \in \partial K_x$ for any point $x \in M$. If, additionally, K_{x_0} is strictly convex at $\phi(x_0)$, then ϕ is a parallel section.

For the classification of minimal maps between Riemannian manifolds we will later use a special case of Theorem A for smooth, symmetric tensors $\phi \in \text{Sym}(E^* \otimes E^*)$. Before stating the result let us recall the following definition due to Hamilton [Ham82, Section 9].

Definition 1.2. (Hamilton). A fiberwise map $\Psi : \text{Sym}(E^* \otimes E^*) \rightarrow \text{Sym}(E^* \otimes E^*)$ is said to satisfy the null-eigenvector condition, if whenever ϑ is a non-negative symmetric 2-tensor at a point $x \in M$ and if $v \in T_x M$ is a null-eigenvector of ϑ , then $\Psi(\vartheta)(v, v) \geq 0$.

The next theorem is the elliptic analogue of the maximum principle of Hamilton [Ham86, Lemma 8.2, p. 174]. More precisely:

Theorem B. Let (M, g_M) be a Riemannian manifold and (E, π, M) a Riemannian vector bundle over M equipped with a metric connection. Suppose that $\phi \in \text{Sym}(E^* \otimes E^*)$ is non-negative definite and satisfies

$$\mathcal{L}\phi + \Psi(\phi) = 0,$$

where here Ψ is a smooth fiberwise map satisfying the null-eigenvector condition. If there is an interior point of M where ϕ has a zero-eigenvalue, then ϕ must have a zero-eigenvalue everywhere. Additionally, if ϕ vanishes identically at an interior point of M , then ϕ vanishes everywhere.

Since the maximum principle for scalar functions has uncountable many applications in Geometric Analysis we expect that the strong maximum principle for sections in vector bundles will have plenty of applications as well. In Section 3 we will apply this strong maximum principle to derive a classification of minimal maps between Riemannian manifolds.

Before stating our results in this direction, let us introduce some new definitions. Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimensions m and n respectively. For any smooth map $f : M \rightarrow N$ its differential df induces a map $\Lambda^k df : \Lambda^k T^*M \rightarrow \Lambda^k T^*N$ given by

$$(\Lambda^k df)(v_1, \dots, v_k) := df(v_1) \wedge \dots \wedge df(v_k),$$

for any smooth vector fields $v_1, \dots, v_k \in TM$. The map $\Lambda^k df$ is called the k -Jacobian of f . The *supremum norm* or the k -dilation $\|\Lambda^k df\|(x)$ of the map f at a point $x \in M$ is defined as the supremum of

$$\sqrt{\det([f^*g_N(v_i, v_j)]_{1 \leq i, j \leq k})}$$

when $\{v_1, \dots, v_m\}$ runs over all orthonormal bases of $T_x M$. The k -dilation measures how much the map stretches k -dimensional volumes. The map $f : M \rightarrow N$ is called *weakly k -volume decreasing* if $\|\Lambda^k df\| \leq 1$, *strictly k -volume decreasing* if $\|\Lambda^k df\| < 1$ and *k -volume preserving* if $\|\Lambda^k df\| = 1$. As usual for $k = 1$ we use the term *length* instead of 1-volume and if $k = 2$ we use the term *area* instead of 2-volume. The map $f : M \rightarrow N$ is called an *isometric immersion*, if $f^*g_N = g_M$. A smooth map $f : M \rightarrow N$ is called *minimal*, if its graph

$$\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}$$

is a minimal submanifold of $(M \times N, g_{M \times N} = g_M \times g_N)$.

One of the main objectives in the present article is to prove the following results:

Theorem C. *Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Suppose M is compact, $m = \dim M \geq 2$ and that there exists a constant $\sigma > 0$ such that the sectional curvatures σ_M of M and σ_N of N and*

the Ricci curvature Ric_M of M satisfy

$$\sigma_M > -\sigma, \quad \frac{1}{m-1} \text{Ric}_M \geq \sigma \geq \sigma_N.$$

If $f : M \rightarrow N$ is a minimal map that is weakly length decreasing, then one of the following holds:

- (i) f is constant.
- (ii) f is an isometric immersion, M is Einstein with $\text{Ric}_M = (m-1)\sigma$ and the restriction of σ_N to $\text{df}(TM)$ is equal to σ .

In particular, any strictly length decreasing minimal map is constant.

A similar statement holds in the case of weakly area decreasing maps.

Theorem D. *Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Suppose M is compact, $m = \dim M \geq 2$ and that there exists a constant $\sigma > 0$ such that the sectional curvatures σ_M of M and σ_N of N and the Ricci curvature Ric_M of M satisfy*

$$\sigma_M > -\sigma, \quad \frac{1}{m-1} \text{Ric}_M \geq \sigma \geq \sigma_N.$$

If $f : M \rightarrow N$ is a smooth minimal map that is weakly area decreasing, then one of the following holds:

- (i) f is constant.
- (ii) There exists a non-empty closed set D such that f is an isometric immersion on D and f is strictly area decreasing on the complement of D . Moreover, M is Einstein on D with $\text{Ric}_M = (m-1)\sigma$ and the restriction of σ_N to $\text{df}(TD)$ is equal to σ .

In particular, any strictly area decreasing minimal map is constant and any area preserving minimal map is an isometric immersion.

In the special case where the manifold N is one-dimensional we have the following stronger theorem:

Theorem E. *Let (M, g_M) and (N, g_N) be two Riemannian manifolds. Suppose that M is compact, $\dim M \geq 2$, $\text{Ric}_M > 0$ and that $\dim N = 1$. Then any smooth minimal map $f : M \rightarrow N$ is constant.*

As pointed out in the final remarks of Section 3.6, Theorems C, D and E are optimal in various ways. We include some examples and remarks concerning the imposed assumptions at the end of the paper.

The paper is organized as follows. In Section 2 we recall the strong maximum principle for uniformly elliptic systems of second order by Weinberger-Evans and give the proofs of Theorems A and B. The geometry of graphs will be treated in Section 3, where we also derive the crucial formula needed in the proof of Theorems C, D and E.

2. STRONG ELLIPTIC MAXIMUM PRINCIPLES FOR SECTIONS IN VECTOR BUNDLES

In this section we shall derive strong elliptic maximum principles for smooth sections in Riemannian vector bundles. The original idea goes back to the fundamental work of Hamilton [Ham82, Ham86] on the Ricci flow, where a strong parabolic maximum principle for symmetric tensors and weak parabolic maximum principles for sections in vector bundles were proven.

2.1. Convex sets. In this subsection we review the basic definitions about the geometry of convex sets in euclidean space such as supporting half-spaces, tangent cones and normal vectors. A brief exposition can be found in the book by Andrews and Hopper [AH11, Appendix B].

Recall that a subset K of \mathbb{R}^n is called *convex* if for any pair of points $z, w \in K$, the segment

$$\mathcal{E}_{z,w} := \{tz + (1-t)w \in \mathbb{R}^n : t \in (0,1)\}$$

is contained in K . The set K is said to be *strictly convex*, if for any pair $z, w \in K$ the segment $\mathcal{E}_{z,w}$ belongs to the interior of K .

A convex set $K \subset \mathbb{R}^n$ may have non-smooth boundary. Hence, there is no well-defined tangent or normal space of K in the classical sense. However, there is a way to generalize these important notions for closed convex subsets of \mathbb{R}^n . This difficulty can be overcome by using the property that points lying outside of the given set can be separated from the set itself by half-spaces. This property, leads to the notion of generalized tangency.

Let K be a closed convex subset of the euclidean space \mathbb{R}^n . A supporting half-space of the set K is a closed half-space of \mathbb{R}^n which contains K and has points of K on its boundary. A supporting hyperplane of K is a hyperplane which is the boundary of a supporting half-space of K . The *tangent cone* $C_{y_0}K$ of K at $y_0 \in \partial K$ is defined as the intersection of all supporting half-spaces of K that contain y_0 .

We may also introduce the notion of normal vectors to the boundary of a closed convex set. Let $K \subset \mathbb{R}^n$ be a closed convex subset and $y_0 \in \partial K$. Then

- (i) A non-zero vector ξ is called *normal vector* of ∂K at y_0 , if ξ is normal to a supporting hyperplane of K passing through y_0 . This normal vector is called *inward pointing*, if it points into the half-space containing the set K .

- (ii) A vector η is called *inward pointing* at $y_0 \in \partial K$, if

$$\langle \xi, \eta \rangle \geq 0$$

for any inward pointing normal vector ξ at y_0 . Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^n .

2.2. Maximum principles for systems. In [Wei75], H. Weinberger established a strong maximum principle for vector valued maps with values in a convex set $K \subset \mathbb{R}^n$ whose boundary ∂K satisfies regularity conditions that he called “*slab conditions*”. Inspired by the ideas of Weinberger, X. Wang in [Wan90] gave a geometric proof of the strong maximum principle, in the case where the boundary of K is of class C^2 . The idea of Wang was to apply the classical maximum principle of Hopf to the function $d(u) : D \rightarrow \mathbb{R}$, whose value at x is equal to the distance of $u(x)$ from the boundary ∂K of K . Very recently, L.C. Evans [Eva10] was able to remove all additional regularity requirements on the boundary of the convex set K by showing that even if $d(u)$ is not twice differentiable, it is still a viscosity super-solution of an appropriate partial differential equation. The argument of Evans is completed by applying a strong maximum principle due to F. Da Lio [DL04] for viscosity super-solutions of partial differential equations.

Theorem 2.1. (Weinberger-Evans). *Let K be a closed, convex set of \mathbb{R}^n and $u : D \subset \mathbb{R}^m \rightarrow K \subset \mathbb{R}^n$ a solution of the uniformly elliptic system of partial differential equations*

$$(\mathcal{L}u)(x) + \Psi(x, u(x)) = 0, \quad x \in D,$$

where D is a domain of \mathbb{R}^m , $\Psi : D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous map that is locally Lipschitz continuous in the second variable and \mathcal{L} is a uniformly elliptic operator given in (). Suppose that*

- (i) *there is a point x_0 in the interior of D such that $u(x_0) \in \partial K$,*
- (ii) *for any $(x, y) \in D \times \partial K$, the vector $\Psi(x, y)$ points into K at the point $y \in \partial K$.*

Then $u(x) \in \partial K$ for any $x \in D$. If ∂K is strictly convex at $u(x_0)$, then u is constant.

Remark 2.2. The above maximum principle is not valid without the convexity assumption. We illustrate this by an example. Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

be the unit open disc in \mathbb{R}^2 and let $h : \partial D \rightarrow \Gamma \subset \mathbb{R}^2$ be a continuous function that maps ∂D onto the upper semicircle

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } y \geq 0\}.$$

Denote now by $u : D \rightarrow \mathbb{R}^2$ the solution of the Dirichlet problem with boundary data given by the function h . Let us examine the image of the harmonic map u . We claim at first that the image of u is contained in the convex hull $\mathcal{C}(\Gamma)$ of the upper semicircle. That is,

$$K := u(\overline{D}) \subset \mathcal{C}(\Gamma) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } y \geq 0\}.$$

Arguing indirectly, let us assume that this is not true. The convex hull $\mathcal{C}(K)$ of K contains $\mathcal{C}(\Gamma)$. Since K is compact, the set $\mathcal{C}(K)$ is also compact. Consequently, there exist a point (x_0, y_0) in D such that $u(x_0, y_0) \in \partial \mathcal{C}(K)$ and $u(x_0, y_0) \notin \partial \mathcal{C}(\Gamma)$. Then, from the maximum principle of Weinberger-Evans we deduce that $u(x, y) \in \partial \mathcal{C}(K)$ for any $(x, y) \in D$. This contradicts with the boundary data imposed by the Dirichlet condition. Therefore, K is contained in $\mathcal{C}(\Gamma)$. From Theorem 2.1, we conclude that there is no common point of K with the x -axes. Hence, K is not convex. The same argument yields that there is no point of D which is mapped to Γ via u . On the other hand, because K is compact, there are infinitely many points of D which are mapped to the boundary of K . Furthermore, we claim that the set K has non-empty interior. In order to show this, suppose to the contrary that $K \setminus \partial K = \emptyset$. Then,

$$\text{rank}(du) \leq 1$$

which implies that the closure of the set $u(D)$ is a continuous curve L joining the points $(-1, 0)$ and $(1, 0)$. But then, the continuity of u leads to a contradiction. Indeed, for any sequence $\{p_k\}_{k \in \mathbb{N}}$ of points of D converging to a point $p \in u^{-1}(0, 1)$, we have $\lim u(p_k) \neq (0, 1)$.

2.3. Maximum principles for sections in vector bundles. Here we give the analogue version of the Weinberger-Evans strong maximum principle for sections in Riemannian vector bundles. Our approach is inspired by ideas developed by Weinberger [Wei75] and Hamilton [Ham82, Ham86].

For the proof of the strong maximum principle we will use a beautiful result due to C. Böhm and B. Wilking [BW07, Lemma 1.2, p. 670].

Lemma 2.3. (Böhm-Wilking). *Suppose that M is a Riemannian manifold and that (E, π, M) is a Riemannian vector bundle over M equipped with a metric connection. Let K be a closed and fiber-convex subset of the bundle E that is invariant under parallel transport. If ϕ is a smooth section with values in K then, for any $x \in M$ and $v \in T_x M$, the Hessian*

$$\nabla_{v,v}^2 \phi = \nabla_v \nabla_v \phi - \nabla_{\nabla_v v} \phi$$

belongs to the tangent cone of K_x at the point $\phi(x)$.

The following result is an immediate consequence of the above lemma.

Lemma 2.4. *Suppose that M is a Riemannian manifold and that (E, π, M) is a Riemannian vector bundle over M equipped with a metric connection. Let K be a closed and fiber-convex subset of E that is invariant under parallel transport. If ϕ is a smooth section with values in K then, for any $x \in M$, the vector $(\mathcal{L}\phi)(x)$ belongs to the tangent cone $C_{\phi(x)} K_x$, for any operator \mathcal{L} of the form given in (**).*

Now we derive the proof of the strong elliptic maximum principle formulated in Theorem A.

Proof of Theorem A. Let $\{\phi_1, \dots, \phi_k\}$ be a geodesic orthonormal frame field of smooth sections in E , defined in a sufficiently small neighborhood U of a local trivialization around $x_0 \in M$. Hence,

$$\phi = \sum_{i=1}^k u_i \phi_i$$

where $u_i : U \rightarrow \mathbb{R}$, $i \in \{1, \dots, k\}$, are smooth functions.

With respect to this frame we have

$$\begin{aligned} \mathcal{L}\phi &= \sum_{i=1}^k \left\{ \mathcal{L}u_i + (\text{gradient terms of } u_i) + \sum_{j=1}^k u_j g_E(\mathcal{L}\phi_j, \phi_i) \right\} \phi_i \\ &= - \sum_{i=1}^k g_E(\Psi(\phi), \phi_i) \phi_i \end{aligned}$$

Therefore, the map $u : U \rightarrow \mathbb{R}^k$, $u = (u_1, \dots, u_k)$, satisfies a uniformly elliptic system of second order of the form

$$\tilde{\mathcal{L}}u + \Phi(u) = 0, \tag{2.1}$$

where here $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$,

$$\Phi := (\Phi_1, \dots, \Phi_k),$$

is given by

$$\Phi_i(u) = g_E \left(\Psi \left(\sum_{j=1}^k u_j \phi_j \right) + \sum_{j=1}^k u_j \mathcal{L} \phi_j, \phi_i \right), \quad (2.2)$$

for any $i \in \{1, \dots, k\}$.

Consider the convex set

$$\mathcal{K} := \{(y_1, \dots, y_k) \in \mathbb{R}^k : \sum_{i=1}^k y_i \phi_i(x_0) \in K_{x_0}\}.$$

Claim 1: *For any point $x \in U$ we have $u(x) \in \mathcal{K}$.*

Indeed, fix a point $x \in U$ and let $\gamma : [0, 1] \rightarrow U$ be the geodesic curve joining the points x and x_0 . Denote by θ the parallel section which is obtained by the parallel transport of $\phi(x)$ along the geodesic γ . Then,

$$\theta \circ \gamma = \sum_{i=1}^k y_i \phi_i \circ \gamma,$$

where $y_i : [0, 1] \rightarrow \mathbb{R}$, $i \in \{1, \dots, k\}$, are smooth functions. Because, θ and ϕ_i , $i \in \{1, \dots, k\}$ are parallel along γ , it follows that

$$0 = \nabla_{\partial_t}(\theta \circ \gamma) = \sum_{i=1}^k y'_i(t) \phi_i(\gamma(t)).$$

Hence, $y_i(t) = y_i(0) = u_i(x)$, for any $t \in [0, 1]$ and $i \in \{1, \dots, k\}$. Therefore,

$$\theta(\gamma(1)) = \theta(x_0) = \sum_{i=1}^k u_i(x) \phi_i(x_0).$$

Since by our assumptions K is invariant under parallel transport, it follows that $\theta(x_0) \in K_{x_0}$. Hence, $u(U) \subset \mathcal{K}$ and this proves Claim 1.

Claim 2: *For any $y \in \partial \mathcal{K}$ the vector $\Phi(y)$ as defined in (2.2) points into \mathcal{K} at y .*

First note that the boundary of each slice K_x is invariant under parallel transport. From (2.2) we deduce that it suffices to prove that both terms appearing on the right hand side of (2.2) point into \mathcal{K} . The first term points into \mathcal{K} by assumption on Ψ . The second term is inward pointing due to Lemma 2.4 by Böhm and Wilking. This completes the proof of Claim 2.

The solution of the uniformly second order elliptic partial differential system (2.1) satisfies all the assumptions of Theorem 2.1. Therefore, because $u(x_0) \in \partial\mathcal{K}$ it follows that $u(U)$ is contained in the boundary $\partial\mathcal{K}$ of \mathcal{K} . Consequently, $\phi(x) \in \partial K$ for any $x \in U$. Since M is connected, we deduce that $\phi(M) \subset \partial K$.

Note, that if \mathcal{K} is additionally strictly convex at $u(x_0)$, then the map u is constant. This implies that

$$\phi(x) = \sum_{i=1}^k u_i(x_0) \phi_i(x)$$

for any $x \in U$. Consequently, ϕ is a parallel section taking all its values in ∂K . This completes the proof of Theorem A. \square

Remark 2.5. Theorem A implies the following: Suppose the fibers of K are cones with vertices at 0 and that they are strictly convex at 0. If $\phi(x) = 0$ in a point $x \in M$, then ϕ vanishes everywhere.

We can now prove Theorem B.

Proof of Theorem B. Let K be the set of all non-negative definite symmetric 2-tensors on M , i.e.

$$K := \{\vartheta \in \text{Sym}(E^* \otimes E^*) : \vartheta \geq 0\}.$$

Each fiber K_x is a closed convex cone that is strictly convex at 0. Moreover, K is invariant under parallel transport. The set of all boundary points of K_x is given by

$$\partial K_x = \{\vartheta \in K_x : \exists \text{ a non-zero } v \in T_x M \text{ such that } \vartheta(v, v) = 0\}.$$

It is a classical fact in Convex Analysis (see for example the book [AH11, Appendix B]), that the tangent cone of K_x at a point ϑ of its boundary is given by

$$C_\vartheta K_x = \{\psi \in \text{Sym}(E_x^* \otimes E_x^*) : \psi(v, v) \geq 0, \forall v \in E_x \text{ with } \vartheta(v, v) = 0\}.$$

Thus ψ is in the tangent cone of K_x at ϑ , if and only if it satisfies the null-eigenvector condition of Hamilton given in Definition 1.2. By Theorem A we immediately get the proof of Theorem B. \square

2.4. A second derivative criterion for symmetric 2-tensors. For $\phi \in \text{Sym}(E^* \otimes E^*)$ a real number λ is called *eigenvalue* of ϕ with respect to g_E at the point $x \in M$, if there exists a non-zero vector $v \in E_x$, such that

$$\phi(v, w) = \lambda g_E(v, w),$$

for any $w \in E_x$. The linear subspace $\text{Eig}_{\lambda, \phi}(x)$ of E_x given by

$$\text{Eig}_{\lambda, \phi}(x) := \{v \in E_x : \phi(v, w) = \lambda g_E(v, w), \text{ for any } w \in E_x\},$$

is called the *eigenspace* of λ at x . Since ϕ is symmetric it admits k real eigenvalues $\lambda_1(x), \dots, \lambda_k(x)$ at each point $x \in M$. We will always arrange the eigenvalues such that $\lambda_1(x) \leq \dots \leq \lambda_k(x)$.

Theorem 2.6. (Second Derivative Criterion) *Let (M, g_M) be a Riemannian manifold and (E, π, M) a Riemannian vector bundle of rank k over the manifold M equipped with a metric connection ∇ . Suppose that $\phi \in \text{Sym}(E^* \otimes E^*)$ is a smooth symmetric 2-tensor. If the biggest eigenvalue λ_k of ϕ admits a local maximum λ at an interior point $x_0 \in M$, then*

$$(\nabla \phi)(v, v) = 0 \quad \text{and} \quad (\mathcal{L} \phi)(v, v) \leq 0,$$

for all vectors v in the eigenspace of λ at x_0 and for all uniformly elliptic second order operators \mathcal{L} .

Remark. Replacing ϕ by $-\phi$ in Theorem 2.6 gives a similar result for the smallest eigenvalue λ_1 of ϕ .

Proof. Let $v \in \text{Eig}_{\lambda, \phi}(x_0)$ be a unit vector and $V \in \Gamma(E)$ a smooth section such that

$$V(x_0) = v \quad \text{and} \quad (\nabla V)(x_0) = 0.$$

Define the symmetric 2-tensor S given by $S := \phi - \lambda g_E$. From our assumptions, the symmetric 2-tensor S is non-positive definite in a small neighborhood of x_0 . Moreover, the biggest eigenvalue of S at x_0 equals 0. Consider the smooth function $f : M \rightarrow \mathbb{R}$, given by

$$f(x) := S(V(x), V(x)).$$

The function f is non-positive in the same neighborhood around x_0 and attains a local maximum in an interior point x_0 . In particular,

$$f(x_0) = 0, \quad df(x_0) = 0 \quad \text{and} \quad (\mathcal{L} f)(x_0) \leq 0.$$

Consider a local orthonormal frame field $\{e_1, \dots, e_m\}$ with respect to g_M defined in a neighborhood of the point x_0 and assume that the expression of \mathcal{L} with respect to this frame is

$$\mathcal{L} = \sum_{i,j=1}^m a^{ij} \nabla_{e_i, e_j}^2 + \sum_{j=1}^m b^j \nabla_{e_j}.$$

A simple calculation yields

$$\nabla_{e_i} f = df(e_i) = (\nabla_{e_i} S)(V, V) + 2S(\nabla_{e_i} V, V).$$

Taking into account that g_E is parallel, we deduce that

$$0 = (\nabla f)(x_0) = (\nabla S)(v, v) = (\nabla \phi)(v, v).$$

Furthermore,

$$\begin{aligned} \nabla_{e_i, e_j}^2 f &= (\nabla_{e_i, e_j}^2 S)(V, V) + 2S(V, \nabla_{e_i, e_j}^2 V) \\ &\quad + 2(\nabla_{e_i} S)(\nabla_{e_j} V, V) + 2(\nabla_{e_j} S)(\nabla_{e_i} V, V) \\ &\quad + 2S(\nabla_{e_i} V, \nabla_{e_j} V). \end{aligned}$$

Bearing in mind the definition of S and using the fact that g_E is parallel with respect to ∇ , we obtain

$$\begin{aligned} \mathcal{L}f &= (\mathcal{L}\phi)(V, V) + 2S(V, \mathcal{L}V) \\ &\quad + \sum_{i,j=1}^m 2a^{ij} \{S(\nabla_{e_i} V, \nabla_{e_j} V) + 2(\nabla_{e_i} S)(\nabla_{e_j} V, V)\} \\ &= (\mathcal{L}\phi)(V, V) + 2S(V, \mathcal{L}V) \\ &\quad + \sum_{i,j=1}^m 2a^{ij} \{S(\nabla_{e_i} V, \nabla_{e_j} V) + 2(\nabla_{e_i} S)(\nabla_{e_j} V, V)\}. \end{aligned}$$

Estimating at x_0 and taking into account that $V(x_0) = v$ is a null eigenvector of S at x_0 , we get

$$0 \geq (\mathcal{L}f)(x_0) = (\mathcal{L}\phi)(v, v).$$

This completes the proof. \square

2.5. An application. In order to demonstrate how to apply the strong elliptic maximum principle and the second derivative criterion, we shall give here an example in the case of hypersurfaces in euclidean space.

Let M be an oriented m -dimensional hypersurface of \mathbb{R}^{m+1} . Denote by ξ a unit normal vector field along the hypersurface. The most natural symmetric 2-tensor on M is the *scalar second fundamental form* h of the hypersurface with respect to the unit normal direction ξ , that is

$$h(v, w) := -\langle d\xi(v), w \rangle,$$

for any $v, w \in TM$. The eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_m$$

of h with respect to the induced metric g are called the *principal curvatures* of the hypersurface. The quantity H given by

$$H := \lambda_1 + \cdots + \lambda_m$$

is called the *scalar mean curvature* of the hypersurface and the function $\|h\|$ given by

$$\|h\|^2 := \lambda_1^2 + \cdots + \lambda_m^2$$

is called the *norm of the second fundamental form* with respect to the metric g . It is well known that if h is non-negative definite, then M is locally the boundary of a convex subset of \mathbb{R}^{m+1} . For this reason, the hypersurface M is called *convex* whenever its scalar second fundamental form is non-negative definite.

In the sequel we will give an alternative short proof of a well-known theorem, first proven by W. Süss [Süs52].

Theorem 2.7. (Süss) *Any closed and convex hypersurface M in \mathbb{R}^{m+1} with constant mean curvature is a round sphere.*

Proof. The Laplacian of the second fundamental form h with respect to the induced Riemannian metric g , is given by Simons' identity [Sim68]

$$\Delta h + \|h\|^2 h - Hh^{(2)} = 0, \quad (2.3)$$

where

$$h^{(2)}(v, w) := \text{trace}(h(v, \cdot) \otimes h(w, \cdot)).$$

Since the manifold M is closed, there exists an interior point $x_0 \in M$, where the smallest principal curvature λ_1 attains its global minimum λ_{\min} . Recall that by convexity we have that $\lambda_{\min} \geq 0$.

The fiberwise map Ψ given by

$$\Psi(\vartheta) = \|\vartheta\|^2 \vartheta - H\vartheta^{(2)},$$

obviously satisfies the null-eigenvector condition.

If $\lambda_{\min} = 0$, then due to Theorem B, the smallest principal curvature of M vanishes everywhere. Hence, $\text{rank } h < m$. It is a well known fact in Differential Geometry that the set

$$M_0 := \{x \in M : \text{rank } h_x = \max_{z \in M} \text{rank } h_z\},$$

is open and dense in M (a standard reference is [Fer71]). From the Codazzi equation, it follows that the nullity distribution

$$\mathcal{D} := \{v \in TM_0 : h(v, w) = 0, \text{ for all } w \in TM_0\},$$

is integrable and its integrals are totally geodesic submanifolds of M . On the other hand, the Gauß formula says that these submanifolds are totally geodesic in \mathbb{R}^{m+1} . Moreover, because M is complete it follows that these submanifolds must be also complete. This contradicts with

the assumption of compactness. Consequently, the minimum λ_{\min} of the smallest principal curvature must be strictly positive.

Let v be a unit eigenvector of h corresponding to λ_{\min} at x_0 . Applying Theorem 2.6, we obtain

$$\begin{aligned} 0 &\geq \|h\|^2(x_0)\lambda_{\min} - H\lambda_{\min}^2 \\ &= \lambda_{\min} (\|h\|^2(x_0) - H\lambda_{\min}), \end{aligned}$$

Because $\|h\|^2 \geq H^2/m$, we deduce that

$$\|h\|^2(x_0) - H\lambda_{\min} \geq H(H/m - \lambda_{\min}) \geq 0.$$

Consequently,

$$0 \geq \lambda_{\min}H(H/m - \lambda_{\min}) \geq 0,$$

and so $H/m = \lambda_{\min}$. On the other hand λ_{\min} is the global minimum of all principal curvatures on M and H is constant. This shows that the smallest principal curvature $\lambda_1(x)$ at an arbitrary point $x \in M$ satisfies

$$\lambda_{\min} \leq \lambda_1(x) \leq H/m = \lambda_{\min}.$$

Therefore M is everywhere umbilic. It is well-known that the only closed and totally umbilic hypersurfaces are the round spheres. \square

3. BERNSTEIN TYPE THEOREMS FOR MINIMAL MAPS

In this section we shall develop the relevant geometric identities for graphs induced by smooth maps $f : M \rightarrow N$. Moreover, we will derive estimates that will be crucial in the proofs of Theorems C, D and E.

According to the Bernstein theorem [Ber27], all complete minimal graphs in the three dimensional euclidean space are generated by affine maps. This result cannot be extended to complete minimal graphs in any euclidean space without imposing further assumptions. There is a very rich and long literature concerning complete minimal graphs which are generated by maps between euclidean spaces, marked by works of W. Fleming [Fle62], S.S. Chern and R. Osserman [CO67], J. Simons [Sim68], E. Bombieri, E. de Giorgi and E. Giusti [BGG69], R. Schoen, L. Simon and S.T. Yau [SSY75], S. Hildebrandt, J. Jost and K.-O. Widmann [HJW80] and many others.

In the last decade there have been obtained several Bernstein type theorems in higher codimension, see for instance [SWX06], [LS10], [HSHV09, HSHV11] and [JXY11].

The generalized Bernstein type problem that we are investigating here is to determine under which additional geometric conditions minimal graphs generated by maps $f : M \rightarrow N$ are totally geodesic. There are several recent results involving mean curvature flow in the case where both M and N are compact. For instance, we mention [Wan01, Wan01b, Wan02], [SW02], [TW04] and [LL11]. In these papers the authors prove that the mean curvature flow of graphs, generated by smooth maps $f : M \rightarrow N$ satisfying suitable conditions, evolves f to a constant map or an isometric immersion as time tends to infinity. This implies in particular Bernstein results for minimal graphs satisfying the same conditions as the initial map.

3.1. Geometry of graphs. Let (M, g_M) and (N, g_N) be Riemannian manifolds of dimension m and n , respectively. The induced metric on the product manifold will be denoted by

$$g_{M \times N} = g_M \times g_N.$$

A smooth map $f : M \rightarrow N$ defines an embedding $F : M \rightarrow M \times N$, by

$$F(x) = (x, f(x)), \quad x \in M.$$

The graph of f is defined to be the submanifold $\Gamma(f) := F(M)$. Since F is an embedding, it induces another Riemannian metric $g := F^*g_{M \times N}$ on M . The two natural projections

$$\pi_M : M \times N \rightarrow M, \quad \pi_N : M \times N \rightarrow N$$

are submersions, that is they are smooth and have maximal rank. Note that the tangent bundle of the product manifold $M \times N$, splits as a direct sum

$$T(M \times N) = TM \oplus TN.$$

The four metrics $g_M, g_N, g_{M \times N}$ and g are related by

$$g_{M \times N} = \pi_M^* g_M + \pi_N^* g_N, \tag{3.1}$$

$$g = F^* g_{M \times N} = g_M + f^* g_N. \tag{3.2}$$

Additionally, let us define the symmetric 2-tensors

$$s_{M \times N} := \pi_M^* g_M - \pi_N^* g_N, \tag{3.3}$$

$$s := F^* s_{M \times N} = g_M - f^* g_N. \tag{3.4}$$

Note that $s_{M \times N}$ is a semi-Riemannian metric of signature (m, k) on the manifold $M \times N$.

The Levi-Civita connection $\nabla^{g_{M \times N}}$ associated to the Riemannian metric $g_{M \times N}$ on $M \times N$ is related to the Levi-Civita connections ∇^{g_M} on (M, g_M) and ∇^{g_N} on (N, g_N) by

$$\nabla^{g_{M \times N}} = \pi_M^* \nabla^{g_M} \oplus \pi_N^* \nabla^{g_N}.$$

The corresponding curvature operator $R_{M \times N}$ on $M \times N$ with respect to the metric $g_{M \times N}$ is related to the curvature operators R_M on (M, g_M) and R_N on (N, g_N) by

$$R_{M \times N} = \pi_M^* R_M \oplus \pi_N^* R_N.$$

Denote the Levi-Civita connection on M with respect to the induced metric $g = F^* g_{M \times N}$ simply by ∇ and the curvature tensor by R .

On the manifold M there are many interesting bundles. The most important one is the *pull-back bundle* $F^*T(M \times N)$. The differential dF of the map F is a section in $F^*T(M \times N) \otimes T^*M$. The covariant derivative of it is called the *second fundamental tensor* A of the graph. That is,

$$A(v, w) := (\tilde{\nabla} dF)(v, w) = \nabla_{dF(v)}^{g_{M \times N}} dF(w) - dF(\nabla_v w)$$

where $v, w \in TM$, $\tilde{\nabla}$ is the induced connection on $F^*T(M \times N) \otimes T^*M$ and ∇ is the Levi-Civita connection associated to the Riemannian metric

$$g := F^* g_{M \times N}.$$

The trace of A with respect to the metric g is called the *mean curvature vector field* of $\Gamma(f)$ and it will be denoted by

$$\vec{H} := \text{trace } A.$$

Note that \vec{H} is a section in the normal bundle of the graph. If \vec{H} vanishes identically the graph is said to be minimal. Following Schoen's [Sch93] terminology, a map $f : M \rightarrow N$ between Riemannian manifolds is called *minimal* if its graph $\Gamma(f)$ is a minimal submanifold of the product space $(M \times N, g_{M \times N})$.

By *Gauß' equation* the curvature tensors R and $R_{M \times N}$ are related by the formula

$$\begin{aligned} R(v_1, w_1, v_2, w_2) &= (F^* R_{M \times N})(v_1, w_1, v_2, w_2) \\ &\quad + g_{M \times N}(A(v_1, v_2), A(w_1, w_2)) \\ &\quad - g_{M \times N}(A(v_1, w_2), A(w_1, v_2)), \end{aligned} \quad (3.5)$$

for any $v_1, v_2, w_1, w_2 \in TM$. Moreover, the second fundamental form satisfies the *Codazzi equation*

$$\begin{aligned} (\nabla_u A)(v, w) - (\nabla_v A)(u, w) &= R_{M \times N}(dF(u), dF(v)) dF(w) \\ &\quad - dF(R(u, v)w), \end{aligned} \quad (3.6)$$

for any u, v, w on TM .

3.2. Singular decomposition. In this subsection we closely follow the notations used in [TW04]. For a fixed point $x \in M$, let

$$\lambda_1^2(x) \leq \cdots \leq \lambda_m^2(x)$$

be the eigenvalues of f^*g_N with respect to g_M . The corresponding values $\lambda_i \geq 0$, $i \in \{1, \dots, m\}$, are usually called *singular values* of the differential df of f and give rise to continuous functions on M . Let

$$r = r(x) = \text{rank } df(x).$$

Obviously, $r \leq \min\{m, n\}$ and $\lambda_1(x) = \cdots = \lambda_{m-r}(x) = 0$. At the point x consider an orthonormal basis $\{\alpha_1, \dots, \alpha_{m-r}; \alpha_{m-r+1}, \dots, \alpha_m\}$ with respect to g_M which diagonalizes f^*g_N . Moreover, at $f(x)$ consider an orthonormal basis $\{\beta_1, \dots, \beta_{n-r}; \beta_{n-r+1}, \dots, \beta_n\}$ with respect to g_N such that

$$df(\alpha_i) = \lambda_i(x) \beta_{n-m+i},$$

for any $i \in \{m-r+1, \dots, m\}$. The above procedure is called the *singular decomposition* of the differential df .

Now we are going to define a special basis for the tangent and the normal space of the graph in terms of the singular values. The vectors

$$e_i := \begin{cases} \alpha_i, & 1 \leq i \leq m-r, \\ \frac{1}{\sqrt{1+\lambda_i^2(x)}} (\alpha_i \oplus \lambda_i(x) \beta_{n-m+i}), & m-r+1 \leq i \leq m, \end{cases} \quad (3.7)$$

form an orthonormal basis with respect to the metric $g_{M \times N}$ of the tangent space $dF(T_x M)$ of the graph $\Gamma(f)$ at x . Moreover, the vectors

$$\xi_i := \begin{cases} \beta_i, & 1 \leq i \leq n-r, \\ \frac{1}{\sqrt{1+\lambda_{i+m-n}^2(x)}} (-\lambda_{i+m-n}(x) \alpha_{i+m-n} \oplus \beta_i), & n-r+1 \leq i \leq n, \end{cases} \quad (3.8)$$

give an orthonormal basis with respect to $g_{M \times N}$ of the normal space $\mathcal{N}_x M$ of the graph $\Gamma(f)$ at the point $f(x)$. From the formulas above, we deduce that

$$s_{M \times N}(e_i, e_j) = \frac{1 - \lambda_i^2(x)}{1 + \lambda_i^2(x)} \delta_{ij}, \quad 1 \leq i, j \leq m. \quad (3.9)$$

Consequently, the eigenvalues of the 2-tensor s with respect to g , are

$$\frac{1 - \lambda_1^2(x)}{1 + \lambda_1^2(x)} \geq \dots \geq \frac{1 - \lambda_{m-1}^2(x)}{1 + \lambda_{m-1}^2(x)} \geq \frac{1 - \lambda_m^2(x)}{1 + \lambda_m^2(x)}.$$

Moreover,

$$s_{M \times N}(\xi_i, \xi_j) = \begin{cases} -\delta_{ij}, & 1 \leq i \leq n - r \\ -\frac{1 - \lambda_{i+m-n}^2(x)}{1 + \lambda_{i+m-n}^2(x)} \delta_{ij}, & n - r + 1 \leq i \leq n. \end{cases} \quad (3.10)$$

and

$$s_{M \times N}(e_{m-r+i}, \xi_{n-r+j}) = -\frac{2\lambda_{m-r+i}(x)}{1 + \lambda_{m-r+i}^2(x)} \delta_{ij}, \quad 1 \leq i, j \leq r. \quad (3.11)$$

3.3. Area decreasing maps. Recall that a map $f : M \rightarrow N$ is *weakly area decreasing* if $\|\Lambda^2 df\| \leq 1$ and *strictly area decreasing* if $\|\Lambda^2 df\| < 1$. The above notions are expressed in terms of the singular values by the inequalities

$$\lambda_i^2(x) \lambda_j^2(x) \leq 1 \quad \text{and} \quad \lambda_i^2(x) \lambda_j^2(x) < 1,$$

for any $1 \leq i < j \leq m$ and $x \in M$, respectively. On the other hand, the sum of two eigenvalues of the tensor s with respect to g equals

$$\frac{1 - \lambda_i^2}{1 + \lambda_i^2} + \frac{1 - \lambda_j^2}{1 + \lambda_j^2} = \frac{2(1 - \lambda_i^2 \lambda_j^2)}{(1 + \lambda_i^2)(1 + \lambda_j^2)}.$$

Hence, the strictly area-decreasing property of the map f is equivalent to the *2-positivity* of the symmetric tensor s .

From the algebraic point of view, the 2-positivity of a symmetric tensor $T \in \text{Sym}(T^*M \otimes T^*M)$ can be expressed as the convexity of another symmetric tensor $T^{[2]} \in \text{Sym}(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$. Indeed, let P and Q be two symmetric 2-tensors. Then, the map $P \oslash Q$ given by

$$\begin{aligned} (P \oslash Q)(v_1 \wedge w_1, v_2 \wedge w_2) &= P(v_1, v_2) Q(w_1, w_2) + P(w_1, w_2) Q(v_1, v_2) \\ &\quad - P(w_1, v_2) Q(v_1, w_2) - P(v_1, w_2) Q(w_1, v_2) \end{aligned}$$

gives rise to an element of $\text{Sym}(\Lambda^2 T^* M \otimes \Lambda^2 T^* M)$. The operator \otimes is called the *Kulkarni-Nomizu product*. Now we assign to each symmetric 2-tensor $T \in \text{Sym}(T^* M \otimes T^* M)$ an element $T^{[2]}$ of the bundle $\text{Sym}(\Lambda^2 T^* M \otimes \Lambda^2 T^* M)$, by setting

$$T^{[2]} := T \otimes g.$$

The Riemannian metric G of the bundle $\Lambda^2 TM$ is related to the Riemannian metric g on the manifold M by the formula

$$G = \frac{1}{2} g \otimes g = \frac{1}{2} g^{[2]}.$$

The relation between the eigenvalues of T and the eigenvalues of $T^{[2]}$ is explained in the following lemma:

Lemma 3.1. *Suppose that T is a symmetric 2-tensor with eigenvalues $\mu_1 \leq \dots \leq \mu_m$ and corresponding eigenvectors $\{v_1, \dots, v_m\}$ with respect to g . Then the eigenvalues of the symmetric 2-tensor $T^{[2]}$ with respect to G are*

$$\mu_i + \mu_j, \quad 1 \leq i < j \leq m,$$

with corresponding eigenvectors

$$v_i \wedge v_j, \quad 1 \leq i < j \leq m.$$

3.4. A Bochner-Weitzenböck formula. Our next goal is to compute the Laplacians of the tensors s and $s^{[2]}$. The next computations closely follow those for a similarly defined tensor in [SW02]. In order to control the smallest eigenvalue of s , let us define the symmetric 2-tensor

$$\Phi_c := s - \frac{1-c}{1+c} g,$$

where c is a non-negative constant.

At first let us compute the covariant derivative of the tensor Φ_c . Since $\nabla g = 0$ and $\nabla^{g_{M \times N}} s_{M \times N} = 0$, we have

$$\begin{aligned} (\nabla_v \Phi_c)(u, w) &= (\nabla_v s)(u, w) \\ &= (\nabla_{\frac{g_{M \times N}}{dF(v)}} s_{M \times N})(dF(u), dF(w)) \\ &\quad + s_{M \times N}(A(v, u), dF(w)) + s_{M \times N}(dF(u), A(v, w)) \\ &= s_{M \times N}(A(v, u), dF(w)) + s_{M \times N}(dF(u), A(v, w)), \end{aligned}$$

for any tangent vectors $u, v, w \in TM$.

Now let us compute the Hessian of Φ_c . Differentiating once more gives

$$\begin{aligned} & (\nabla_{v_1, v_2}^2 \Phi_c)(u, w) \\ &= s_{M \times N}((\nabla_{v_1} A)(v_2, u), dF(w)) + s_{M \times N}(A(v_2, u), A(v_1, w)) \\ & \quad + s_{M \times N}(A(v_1, u), A(v_2, w)) + s_{M \times N}(dF(u), (\nabla_{v_1} A)(v_2, w)), \end{aligned}$$

for any tangent vectors $v_1, v_2, u, w \in TM$. Applying Codazzi's equation (3.6) and exploiting the symmetry of A and $s_{M \times N}$, we derive

$$\begin{aligned} & (\nabla_{v_1, v_2}^2 \Phi_c)(u, w) \\ &= s_{M \times N}((\nabla_u A)(v_1, v_2) + R_{M \times N}(dF(v_1), dF(u)) dF(v_2), dF(w)) \\ & \quad + s_{M \times N}((\nabla_w A)(v_1, v_2) + R_{M \times N}(dF(v_1), dF(w)) dF(v_2), dF(u)) \\ & \quad + s_{M \times N}(A(v_1, u), A(v_2, w)) + s_{M \times N}(A(v_1, w), A(v_2, u)) \\ & \quad - s(R(v_1, u)v_2, w) - s(R(v_1, w)v_2, u). \end{aligned}$$

The decomposition of the tensors $s_{M \times N}$ and $R_{M \times N}$, implies

$$\begin{aligned} & s_{M \times N}(R_{M \times N}(dF(v_1), dF(u)) dF(v_2), dF(w)) \\ &= (\pi_M^* g_M)(R_{M \times N}(dF(v_1), dF(u)) dF(v_2), dF(w)) \\ & \quad - (\pi_N^* g_N)(R_{M \times N}(dF(v_1), dF(u)) dF(v_2), dF(w)) \\ &= g_M(R_M(v_1, u)v_2, w) - g_N(R_N(df(v_1), df(u)) df(v_2), df(w)) \\ &= R_N(df(v_1), df(u), df(v_2), df(w)) - R_M(v_1, u, v_2, w). \end{aligned}$$

Gauß' equation (3.5) gives

$$\begin{aligned} & -s(R(v_1, u)v_2, w) \\ &= -\Phi_c(R(v_1, u)v_2, w) - \frac{1-c}{1+c} g(R(v_1, u)v_2, w) \\ &= -\Phi_c(R(v_1, u)v_2, w) + \frac{1-c}{1+c} R(v_1, u, v_2, w) \\ &= -\Phi_c(R(v_1, u)v_2, w) \\ & \quad + \frac{1-c}{1+c} \{g_{M \times N}(A(v_1, v_2), A(u, w)) - g_{M \times N}(A(v_1, w), A(v_2, u))\} \\ & \quad + \frac{1-c}{1+c} R_M(v_1, u, v_2, w) + \frac{1-c}{1+c} R_N(df(v_1), df(u), df(v_2), df(w)). \end{aligned}$$

In the sequel consider any local orthonormal frame field $\{e_1, \dots, e_m\}$ with respect to the induced metric g on M . Then, taking a trace, we derive the Laplacian of the tensor Φ_c .

Let us now summarize the previous computations in the next lemma:

Lemma 3.2. *For any smooth map $f : M \rightarrow N$, the symmetric tensor Φ_c satisfies the identity*

$$\begin{aligned}
(\Delta \Phi_c)(v, w) &= s_{M \times N}(\nabla_v \vec{H}, dF(w)) + s_{M \times N}(\nabla_w \vec{H}, dF(v)) \\
&\quad + 2 \frac{1-c}{1+c} g_{M \times N}(\vec{H}, A(v, w)) \\
&\quad + \Phi_c(\text{Ric } v, w) + \Phi_c(\text{Ric } w, v) \\
&\quad + 2 \sum_{k=1}^m \left(s_{M \times N} - \frac{1-c}{1+c} g_{M \times N} \right) (A(e_k, v), A(e_k, w)) \\
&\quad + \frac{4}{1+c} \sum_{k=1}^m \left(f^* R_N(e_k, v, e_k, w) - c R_M(e_k, v, e_k, w) \right),
\end{aligned}$$

where

$$\text{Ric } v := - \sum_{k=1}^m R(e_k, v) e_k$$

is the Ricci operator on (M, g) and $\{e_1, \dots, e_m\}$ is any orthonormal frame with respect to the induced metric g .

The expressions of the covariant derivative and the Laplacian of a symmetric 2-tensor $T^{[2]}$ are given in the following Lemma. The proof follows by a straightforward computation and for that reason will be omitted.

Lemma 3.3. *Any symmetric 2-tensor T satisfies the identities,*

$$(i) \quad \nabla_v T^{[2]} = (\nabla_v T)^{[2]},$$

$$(ii) \quad \nabla_{v,v}^2 T^{[2]} = (\nabla_{v,v}^2 T)^{[2]},$$

$$(iii) \quad \Delta T^{[2]} = (\Delta T)^{[2]},$$

for any vector v on the manifold M .

3.5. Proofs of the Theorems C, D and E. We will first show the following lemma.

Lemma 3.4. *Let $f : M \rightarrow N$ be weakly length decreasing. Suppose $\{e_1, \dots, e_m\}$ is orthonormal with respect to g such that it diagonalizes*

the tensor s . Then for any e_l we have

$$\begin{aligned}
& 2 \sum_{k=1}^m \left(R_M(e_k, e_l, e_k, e_l) - f^* R_N(e_k, e_l, e_k, e_l) \right) \\
&= 2 \sum_{k \neq l} \frac{\lambda_k^2}{1 + \lambda_k^2} \left\{ (\sigma - \sigma_N(df(e_k) \wedge df(e_l))) f^* g_N(e_l, e_l) \right. \\
&\quad \left. + \sigma(g_M(e_l, e_l) - f^* g_N(e_l, e_l)) \right\} \\
&\quad + \text{Ric}_M(e_l, e_l) - (m-1) \sigma g_M(e_l, e_l) \\
&\quad + \sum_{k \neq l} \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \left(\sigma_M(e_k \wedge e_l) + \sigma \right) g_M(e_l, e_l),
\end{aligned}$$

where Ric_M denotes the Ricci curvature with respect to g_M , $\sigma_M(e_k \wedge e_l)$ and $\sigma_N(df(e_k) \wedge df(e_l))$ are the sectional curvatures of the planes $e_k \wedge e_l$ on (M, g_M) and $df(e_k) \wedge df(e_l)$ on (N, g_N) respectively.

Proof. In terms of the singular values we get

$$s(e_k, e_k) = g_M(e_k, e_k) - f^* g_N(e_k, e_k) = \frac{1 - \lambda_k^2}{1 + \lambda_k^2}.$$

Since

$$1 = g(e_k, e_k) = g_M(e_k, e_k) + f^* g_N(e_k, e_k)$$

we derive

$$g_M(e_k, e_k) = \frac{1}{1 + \lambda_k^2}, \quad f^* g_N(e_k, e_k) = \frac{\lambda_k^2}{1 + \lambda_k^2}$$

and

$$2g_M(e_k, e_k) = \frac{1 - \lambda_k^2}{1 + \lambda_k^2} + 1, \quad -2f^* g_N(e_k, e_k) = \frac{1 - \lambda_k^2}{1 + \lambda_k^2} - 1.$$

Note also that for any $k \neq l$ we have

$$g_M(e_k, e_l) = f^* g_N(e_k, e_l) = g(e_k, e_l) = 0.$$

We compute

$$\begin{aligned}
& 2 \sum_{k=1}^m \left(R_M(e_k, e_l, e_k, e_l) - f^* R_N(e_k, e_l, e_k, e_l) \right) \\
&= 2 \sum_{k \neq l} \sigma_M(e_k \wedge e_l) g_M(e_k, e_k) g_M(e_l, e_l) \\
&\quad - 2 \sum_{k \neq l} \sigma_N(df(e_k) \wedge df(e_l)) f^* g_N(e_k, e_k) f^* g_N(e_l, e_l).
\end{aligned}$$

Hence the formula for $g_M(e_k, e_k)$ implies

$$\begin{aligned}
& 2 \sum_{k=1}^m \left(R_M(e_k, e_l, e_k, e_l) - f^* R_N(e_k, e_l, e_k, e_l) \right) \\
&= \sum_{k \neq l} \left(1 + \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \right) \sigma_M(e_k \wedge e_l) g_M(e_l, e_l) \\
&\quad + 2 \sum_{k \neq l} f^* g_N(e_k, e_k) \left\{ (\sigma - \sigma_N(df(e_k) \wedge df(e_l))) f^* g_N(e_l, e_l) \right. \\
&\quad \quad \left. + \sigma(g_M(e_l, e_l) - f^* g_N(e_l, e_l)) \right\} \\
&\quad - 2\sigma \sum_{k \neq l} f^* g_N(e_k, e_k) g_M(e_l, e_l) \\
&= \sum_{k \neq l} \left(1 + \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \right) \sigma_M(e_k \wedge e_l) g_M(e_l, e_l) \\
&\quad + 2 \sum_{k \neq l} f^* g_N(e_k, e_k) \left\{ (\sigma - \sigma_N(df(e_k) \wedge df(e_l))) f^* g_N(e_l, e_l) \right. \\
&\quad \quad \left. + \sigma(g_M(e_l, e_l) - f^* g_N(e_l, e_l)) \right\} \\
&\quad + \sigma \sum_{k \neq l} \left(\frac{1 - \lambda_k^2}{1 + \lambda_k^2} - 1 \right) g_M(e_l, e_l).
\end{aligned}$$

We may then continue to get

$$\begin{aligned}
& 2 \sum_{k=1}^m \left(R_M(e_k, e_l, e_k, e_l) - f^* R_N(e_k, e_l, e_k, e_l) \right) \\
&= 2 \sum_{k \neq l} f^* g_N(e_k, e_k) \left\{ (\sigma - \sigma_N(df(e_k) \wedge df(e_l))) f^* g_N(e_l, e_l) \right. \\
&\quad \quad \left. + \sigma(g_M(e_l, e_l) - f^* g_N(e_l, e_l)) \right\} \\
&\quad + \text{Ric}_M(e_l, e_l) - (m-1)\sigma g_M(e_l, e_l) \\
&\quad + \sum_{k \neq l} \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \left(\sigma_M(e_k \wedge e_l) + \sigma \right) g_M(e_l, e_l).
\end{aligned}$$

This completes the proof. \square

Proof of Theorem C. Suppose that $f : M \rightarrow N$ is weakly length decreasing. Then the tensor s satisfies

$$s = g_M - f^* g_N \geq 0.$$

In case $s > 0$ the map f is also strictly area decreasing. Thus in such a case the statement follows from Theorem D which we will prove further below. It remains to show that s vanishes identically, if s admits a null-eigenvalue somewhere.

Claim 1. *The tensor s has a null-eigenvalue everywhere on M , if this is the case in at least one point $x \in M$.*

Since $s = \Phi_1$, from Lemma 3.2 we get

$$\Delta s + \Psi(s) = 0,$$

where

$$\begin{aligned} (\Psi(\vartheta))(v, w) &= -\vartheta(\text{Ric } v, w) - \vartheta(\text{Ric } w, v) \\ &\quad - 2 \sum_{k=1}^m s_{M \times N}(A(e_k, v), A(e_k, w)) \\ &\quad + 2 \sum_{k=1}^m \left(R_M(e_k, v, e_k, w) - f^* R_N(e_k, v, e_k, w) \right). \end{aligned}$$

Let v be a null-eigenvector of the symmetric, positive semi-definite tensor ϑ . Since f is weakly length decreasing, equation (3.10) shows that $s_{M \times N}$ is non-positive definite on the normal bundle of the graph. Hence

$$(\Psi(\vartheta))(v, v) \geq 2 \sum_{k=1}^m \left(R_M(e_k, v, e_k, v) - f^* R_N(e_k, v, e_k, v) \right) \geq 0,$$

where we have used Lemma 3.4 and the curvature assumptions on (M, g_M) , (N, g_N) respectively. This shows that Ψ satisfies the null-eigenvector condition and Claim 1 follows from the strong maximum principle in Theorem B.

Claim 2. *If s admits a null-eigenvalue at some point $x \in M$, then s vanishes at x .*

We already know that the tensor s admits a null-eigenvalue everywhere on M . Since $s \geq 0$ we may then apply the test criterion Theorem 2.6 to the tensor s at an arbitrary point $x \in M$. At x consider a basis $\{e_1, \dots, e_m\}$, orthonormal with respect to g consisting of eigenvectors of s , such that $v := e_m$ is a null-eigenvector of s and $\lambda_m^2 = 1$. From

Lemma 3.4, we conclude

$$\begin{aligned}
0 &\geq (\Psi(s))(e_m, e_m) \\
&\geq 2 \sum_{k \neq m} \frac{\lambda_k^2}{1 + \lambda_k^2} \left\{ \underbrace{(\sigma - \sigma_N(df(e_k) \wedge df(e_m)))}_{\geq 0} \underbrace{f^*g_N(e_l, e_l)}_{\geq 0} \right. \\
&\quad \left. + \sigma \underbrace{(g_M(e_m, e_m) - f^*g_N(e_m, e_m))}_{=0} \right\} \\
&\quad + \underbrace{\text{Ric}_M(e_m, e_m) - (m-1)\sigma g_M(e_m, e_m)}_{\geq 0} \\
&\quad + \sum_{k \neq m} \underbrace{\frac{1 - \lambda_k^2}{1 + \lambda_k^2}}_{\geq 0} \underbrace{(\sigma_M(e_k \wedge e_m) + \sigma)}_{> 0} \underbrace{g_M(e_m, e_m)}_{=\frac{1}{2}} = 0, \tag{3.12}
\end{aligned}$$

because the curvature assumptions on (M, g_M) and (N, g_N) imply that the right hand side is a sum of non-negative terms and thus we conclude that all of them must vanish. In particular,

$$\sigma_M(e_k \wedge e_m) + \sigma > 0$$

implies $\lambda_k^2 = 1$ for all k .

Now we can finish the proof of Theorem C. Claim 1 and 2 imply that a weakly length decreasing map f which is not strictly length decreasing must be an isometric immersion. Once we know that all tangent vectors at x are null-eigenvectors of s , we may choose e_m in (3.12) arbitrarily. Then

$$\text{Ric}_M(v, v) = (m-1)\sigma g_M(v, v)$$

and

$$\sigma = \sigma_N(df(v), df(w))$$

for all linearly independent vectors $v, w \in T_x M$. This completes the proof of Theorem C. \square

Proof of Theorem D. Since the manifold M is compact, there exists a point x_0 where the smallest eigenvalue of $s^{[2]}$ with respect to the metric G attains its minimum. Let us denote this value by ρ_0 . Note that in terms of the singular values

$$\lambda_1^2 \leq \dots \leq \lambda_m^2$$

we must have

$$\rho_0 = \frac{1 - \lambda_m^2(x_0)}{1 + \lambda_m^2(x_0)} + \frac{1 - \lambda_{m-1}^2(x_0)}{1 + \lambda_{m-1}^2(x_0)} \geq 0.$$

For simplicity we set

$$\kappa := \lambda_{m-1}^2(x_0) \quad \text{and} \quad \mu := \lambda_m^2(x_0).$$

Hence,

$$\rho_0 = 2 \frac{1 - \kappa\mu}{(1 + \kappa)(1 + \mu)}.$$

Claim 3. *If $\mu = 0$, then the map f is constant.*

In this case we have $\rho_0 = 2$. Because ρ_0 is the minimum of the smallest eigenvalue of the symmetric tensor $s^{[2]}$, we obtain

$$1 \leq \frac{1 - \lambda_i^2(x)\lambda_j^2(x)}{(1 + \lambda_i^2(x))(1 + \lambda_j^2(x))},$$

for any $x \in M$ and $1 \leq i < j \leq m$. From the above inequality one can readily see that all the singular values of f vanish everywhere. Thus, in this case f is constant. This completes the proof of Claim 3.

Since we are assuming that f is weakly area decreasing, we deduce that $\kappa\mu \leq 1$. Consider now the symmetric 2-tensor

$$\Phi := \Phi_{\frac{2-\rho_0}{2+\rho_0}} = s - \frac{\rho_0}{2} g.$$

According to Lemma 3.3,

$$\Delta\Phi^{[2]} = (\Delta\Phi)^{[2]}.$$

At x_0 consider an orthonormal bases $\{e_1, \dots, e_m\}$ with respect to g such that s becomes diagonal and

$$s(e_k, e_k) = \frac{1 - \lambda_k^2}{1 + \lambda_k^2}.$$

According to Theorem 2.6, we obtain

$$\begin{aligned} 0 &\leq (\Delta\Phi)^{[2]}(e_{m-1} \wedge e_m, e_{m-1} \wedge e_m) \\ &= (\Delta\Phi)(e_{m-1}, e_{m-1}) + (\Delta\Phi)(e_m, e_m). \end{aligned}$$

In view of Lemma 3.2, we deduce that

$$\begin{aligned}
0 \leq & 2\Phi(\text{Ric } e_{m-1}, e_{m-1}) + 2\Phi(\text{Ric } e_m, e_m) \\
& + 2 \sum_{k=1}^m \left(s_{M \times N} - \frac{\rho_0}{2} g_{M \times N} \right) (A(e_k, e_{m-1}), A(e_k, e_{m-1})) \\
& + 2 \sum_{k=1}^m \left(s_{M \times N} - \frac{\rho_0}{2} g_{M \times N} \right) (A(e_k, e_m), A(e_k, e_m)) \\
& + (2 + \rho_0) \sum_{k=1}^m f^* R_N(e_k, e_{m-1}, e_k, e_{m-1}) \\
& - (2 - \rho_0) \sum_{k=1}^m R_M(e_k, e_{m-1}, e_k, e_{m-1}) \\
& + (2 + \rho_0) \sum_{k=1}^m f^* R_N(e_k, e_m, e_k, e_m) \\
& - (2 - \rho_0) \sum_{k=1}^m R_M(e_k, e_m, e_k, e_m). \tag{3.13}
\end{aligned}$$

Because e_m is an eigenvector of s with respect to g , we have

$$\Phi(\text{Ric } e_m, e_m) = \frac{\kappa - \mu}{(1 + \kappa)(1 + \mu)} g(\text{Ric } e_m, e_m).$$

From the Gauß equation (3.5) and the minimality of the graph, we obtain that

$$\begin{aligned}
g(\text{Ric } e_m, e_m) &= \sum_{k=1}^m R_M(e_k, e_m, e_k, e_m) \\
&+ \sum_{k=1}^m f^* R_N(e_k, e_m, e_k, e_m) \\
&- \sum_{k=1}^m g_{M \times N}(A(e_k, e_m), A(e_k, e_m)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi(\text{Ric } e_m, e_m) &= \frac{\kappa - \mu}{(1 + \kappa)(1 + \mu)} \sum_{k=1}^m R_M(e_k, e_m, e_k, e_m) \\
&+ \frac{\kappa - \mu}{(1 + \kappa)(1 + \mu)} \sum_{k=1}^m f^* R_N(e_k, e_m, e_k, e_m) \\
&- \frac{\kappa - \mu}{(1 + \kappa)(1 + \mu)} \sum_{k=1}^m g_{M \times N}(A(e_k, e_m), A(e_k, e_m)). \tag{3.14}
\end{aligned}$$

Similarly,

$$\begin{aligned} \Phi(\text{Ric } e_{m-1}, e_{m-1}) &= \frac{\mu-\kappa}{(1+\kappa)(1+\mu)} \sum_{k=1}^m R_M(e_k, e_{m-1}, e_k, e_{m-1}) \\ &+ \frac{\mu-\kappa}{(1+\kappa)(1+\mu)} \sum_{k=1}^m f^* R_N(e_k, e_{m-1}, e_k, e_{m-1}) \\ &- \frac{\mu-\kappa}{(1+\kappa)(1+\mu)} \sum_{k=1}^m g_{M \times N}(A(e_k, e_{m-1}), A(e_k, e_{m-1})). \end{aligned} \quad (3.15)$$

In view of (3.14) and (3.15), the inequality (3.13) can be now written equivalently in the form

$$\begin{aligned} 0 &\leq \sum_{k=1}^m (s_{M \times N} - \frac{1-\mu}{1+\mu} g_{M \times N})(A(e_k, e_m), A(e_k, e_m)) \\ &+ \sum_{k=1}^m (s_{M \times N} - \frac{1-\kappa}{1+\kappa} g_{M \times N})(A(e_k, e_{m-1}), A(e_k, e_{m-1})) \\ &+ \frac{2}{1+\mu} \sum_{k=1}^m (f^* R_N - \mu R_M)(e_k, e_m, e_k, e_m) \\ &+ \frac{2}{1+\kappa} \sum_{k=1}^m (f^* R_N - \kappa R_M)(e_k, e_{m-1}, e_k, e_{m-1}). \end{aligned} \quad (3.16)$$

Claim 4. *The sum \mathcal{A} of the first two terms on the right hand side of inequality (3.16) is non-positive.*

Indeed, if $\mu = 0$, then f is constant by Claim 3 and thus $\mathcal{A} = 0$. So, let us consider the case where $\mu > 0$. From Theorem 2.6 again, we have

$$\begin{aligned} 0 &= (\nabla_{e_k}(s^{[2]} - \rho_0 G))(e_m \wedge e_{m-1}, e_m \wedge e_{m-1}) \\ &= 2(\nabla_{e_k} s)(e_m, e_m) + 2(\nabla_{e_k} s)(e_{m-1}, e_{m-1}) \\ &= 4s_{M \times N}(A(e_k, e_m), e_m) + 4s_{M \times N}(A(e_k, e_{m-1}), e_{m-1}) \end{aligned} \quad (3.17)$$

for any k . Since $\dim(N) = 1$ implies that $\text{rank}(df) \leq 1$, from (3.11) we obtain

$$\begin{aligned} 0 &= A_{\xi_n}(e_k, e_m) s_{M \times N}(\xi_n, e_m) + A_{\xi_n}(e_k, e_{m-1}) \underbrace{s_{M \times N}(\xi_n, e_{m-1})}_{=0} \\ &= -2A_{\xi_n}(e_k, e_m) \underbrace{\frac{\sqrt{\mu}}{1+\mu}}_{>0}, \end{aligned}$$

where here

$$A_{\xi}(v, w) := g_N(A(v, w), \xi), \quad v, w \in T_x M,$$

stands for the second fundamental form of the graph $\Gamma(f)$ in the normal direction ξ and the normal basis $\{\xi_1, \dots, \xi_n\}$ is chosen as in (3.10). Hence, by the weakly area decreasing property of f , we get

$$\mathcal{A} = - \sum_{k=1}^m \left(\frac{1-\mu}{1+\mu} + \frac{1-\kappa}{1+\kappa} \right) A_{\xi_n}^2(e_k, e_{m-1}) \leq 0.$$

In case $\dim(N) \geq 2$, from (3.10), (3.17) and the weakly area decreasing condition we obtain

$$\begin{aligned} \mathcal{A} &= \sum_{k=1}^m (s_{M \times N} - \frac{1-\mu}{1+\mu} g_{M \times N})(A(e_k, e_m), A(e_k, e_m)) \\ &\quad + \sum_{k=1}^m (s_{M \times N} - \frac{1-\kappa}{1+\kappa} g_{M \times N})(A(e_k, e_{m-1}), A(e_k, e_{m-1})) \\ &\leq -2 \frac{1-\mu}{1+\mu} \sum_{k=1}^m A_{\xi_n}^2(e_k, e_m) - 2 \frac{1-\kappa}{1+\kappa} \sum_{k=1}^m A_{\xi_{n-1}}^2(e_k, e_{m-1}). \end{aligned}$$

In view of equations (3.17) and (3.11), we have

$$\begin{aligned} 0 &= s_{M \times N}(A(e_k, e_m), e_m) + s_{M \times N}(A(e_k, e_{m-1}), e_{m-1}) \\ &= -2 \frac{\sqrt{\mu}}{1+\mu} A_{\xi_n}(e_k, e_m) - 2 \frac{\sqrt{\kappa}}{1+\kappa} A_{\xi_{n-1}}(e_k, e_{m-1}). \end{aligned}$$

Hence,

$$A_{\xi_n}^2(e_k, e_m) = \frac{\kappa(1+\mu)^2}{\mu(1+\kappa)^2} A_{\xi_{n-1}}^2(e_k, e_{m-1}).$$

Because, $\kappa \leq \mu$ and $\kappa\mu \leq 1$, we deduce that

$$\frac{\kappa(1+\mu)^2}{\mu(1+\kappa)^2} \leq 1.$$

This proves our assertion. Now it is clear that the quantity \mathcal{A} is always non-positive which proves Claim 4.

Claim 5. *The sum \mathcal{B} of the last two terms on the right hand side of inequality (3.16) is non-positive.*

We have,

$$\begin{aligned} \mathcal{B} &= \frac{1}{1+\mu} \underbrace{\sum_{k=1}^m 2(f^*R_N - \mu R_M)(e_k, e_m, e_k, e_m)}_{=: \mathcal{B}_1} \\ &\quad + \frac{1}{1+\kappa} \underbrace{\sum_{k=1}^m 2(f^*R_N - \kappa R_M)(e_k, e_{m-1}, e_k, e_{m-1})}_{=: \mathcal{B}_2} \end{aligned}$$

From the identities (3.2) and (3.4), we deduce that

$$g_M = \frac{1}{2}(g+s) \quad \text{and} \quad f^*g_N = \frac{1}{2}(g-s).$$

Since $\{e_1, \dots, e_m\}$ diagonalizes g and s , it follows that it diagonalizes g_M and f^*g_N as well. In fact, for any $i \in \{1, \dots, m\}$, we have

$$f^*g_N(e_i, e_i) = \lambda_i^2 g_M(e_i, e_i).$$

Proceeding exactly as in the proof of Lemma 3.4, but using $\mu\sigma_M$ instead of σ_M and $\mu\sigma$ instead of σ , we obtain that

$$\begin{aligned} \mathcal{B}_1 &= 2 \sum_{k \neq m} \left(f^*R_N(e_k, e_m, e_k, e_m) - \mu R_M(e_k, e_m, e_k, e_m) \right) \\ &= -\frac{2\mu}{1+\mu} \sum_{k \neq m} f^*g_N(e_k, e_k) \left(\sigma - \sigma_N(df(e_k) \wedge df(e_m)) \right) \\ &\quad - \mu \left(\text{Ric}_M(e_m, e_m) - (m-1)\sigma g_M(e_m, e_m) \right) \\ &\quad - \frac{\mu}{1+\mu} \sum_{k \neq m} \frac{1-\lambda_k^2}{1+\lambda_k^2} \left(\sigma_M(e_k \wedge e_m) + \sigma \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{B}_2 &= 2 \sum_{k \neq m-1} \left(f^*R_N(e_k, e_{m-1}, e_k, e_{m-1}) - \kappa R_M(e_k, e_{m-1}, e_k, e_{m-1}) \right) \\ &= -\frac{2\kappa}{1+\kappa} \sum_{k \neq m-1} f^*g_N(e_k, e_k) \left(\sigma - \sigma_N(df(e_k) \wedge df(e_{m-1})) \right) \\ &\quad - \kappa \left(\text{Ric}_M(e_{m-1}, e_{m-1}) - (m-1)\sigma g_M(e_{m-1}, e_{m-1}) \right) \\ &\quad - \frac{\kappa}{1+\kappa} \sum_{k \neq m-1} \frac{1-\lambda_k^2}{1+\lambda_k^2} \left(\sigma_M(e_k \wedge e_{m-1}) + \sigma \right). \end{aligned}$$

Taking into account that $\lambda_1^2 \leq \dots \leq \lambda_{m-2}^2 \leq 1$, we deduce that

$$\begin{aligned}
\mathcal{B} &= -\frac{2\mu}{(1+\mu)^2} \sum_{k \neq m} f^* g_N(e_k, e_k) \left(\sigma - \sigma_N(df(e_k) \wedge df(e_m)) \right) \\
&\quad -\frac{2\kappa}{(1+\kappa)^2} \sum_{k \neq m-1} f^* g_N(e_k, e_k) \left(\sigma - \sigma_N(df(e_k) \wedge df(e_{m-1})) \right) \\
&\quad -\frac{\mu}{1+\mu} \left(\text{Ric}_M(e_m, e_m) - (m-1)\sigma g_M(e_m, e_m) \right) \\
&\quad -\frac{\kappa}{1+\kappa} \left(\text{Ric}_M(e_{m-1}, e_{m-1}) - (m-1)\sigma g_M(e_{m-1}, e_{m-1}) \right) \\
&\quad -\frac{\mu}{(1+\mu)^2} \sum_{k=1}^{m-2} \frac{1-\lambda_k^2}{1+\lambda_k^2} \left(\sigma_M(e_k \wedge e_m) + \sigma \right) \\
&\quad -\frac{\kappa}{(1+\kappa)^2} \sum_{k=1}^{m-2} \frac{1-\lambda_k^2}{1+\lambda_k^2} \left(\sigma_M(e_k \wedge e_m) + \sigma \right) \\
&\quad -\frac{(\kappa+\mu)(1-\kappa\mu)}{(1+\kappa)(1+\mu)} \left(\sigma_M(e_{m-1} \wedge e_m) + \sigma \right) \\
&\leq 0.
\end{aligned} \tag{3.18}$$

This completes the proof of Claim 5.

Now we shall distinguish two cases.

Case 1. Assume at first that f is strictly area decreasing. Hence, $\kappa\mu < 1$. In view of our curvature assumptions, Claim 4, Claim 5, (3.18) and from inequality (3.16) we deduce that $\kappa = \mu = 0$. Hence, from Claim 3 the map f must be constant.

Case 2. Suppose now that there exists a point $x_0 \in M$ such that $\|\Lambda^2 df\|(x_0) = 1$. In this case we have that $\kappa\mu = 1$. From Claim 1, (3.18) and inequality (3.16) we deduce that at x_0 we must have

$$1 = \lambda_1^2(x_0) = \dots = \lambda_{m-2}^2(x_0) \leq \kappa \leq 1.$$

Hence, $\kappa = 1$ and so $\mu = 1$. Therefore, at each point x where $\Phi^{[2]}$ has a zero eigenvalue, all the singular values of f are equal to 1. Thus, the set

$$D := \{x \in M : \|\Lambda^2 df\| = 1\},$$

is closed, non-empty and moreover $D = \{x \in M : f^*g_N = g_M\}$. Obviously, the map f is strictly area decreasing on the complement of D . Moreover, by (3.18), $\text{Ric}_M = (m-1)\sigma$ at any point of D and the restriction of σ_N to $df(TD)$ is equal to σ .

This completes the proof of Theorem D. \square

Proof of Theorem E. Note that in this case the singular values of the map f are

$$0 = \lambda_1^2 = \cdots = \lambda_{m-1}^2 = \kappa \leq \mu.$$

Hence, automatically, f is strictly area decreasing. From Claim 4, Claim 5, inequality (3.13) and (3.18), we deduce that

$$0 \leq -2\mu \operatorname{Ric}_M(e_m, e_m) \leq 0.$$

Thus $\mu = 0$ and f is a constant map. This completes the proof of Theorem E. \square

3.6. Final remarks. We end this paper with examples and remarks concerning the imposed assumptions in Theorems C, D and E.

Remark 3.5. In several cases, graphical submanifolds over (M, g_M) with *parallel mean curvature*, i.e.,

$$\nabla^\perp H = 0,$$

where ∇^\perp stands for the connection of the normal bundle, must be minimal. This problem was first considered by Chern in [Che65]. So, whenever graphs with parallel mean curvature vector are minimal we can immediately apply Theorems C, D and E. For example this can be done for graphs considered in the paper by G. Li and I.M.C. Salavessa [LS10].

Remark 3.6. The reason that the result of Theorem D is weaker than that of Theorem C is due to the fact that in Theorem D we cannot apply the strong elliptic maximum principle stated in Theorem B. In fact, the null-eigenvector condition of the corresponding tensor $\Psi(\vartheta^{[2]})$ in the equation of $\Delta s^{[2]}$ seems to hold only for some weakly 2-positive definite tensors ϑ , including s .

Remark 3.7. In some situations, a minimal map $f : M \rightarrow N$ satisfying the assumptions in Theorem D can only be constant. For instance, if $\dim M > \dim N$ the map f cannot be an isometric immersion since $\operatorname{rank}(df) < \dim M$. Moreover, if M is not Einstein or the sectional curvature of N is strictly less than σ , then any such map must be constant.

Remark 3.8. In this remark we show that the assumptions on the curvatures of M and N in Theorems C and D are sharp.

- i) **Scaling.** Suppose that $f : M \rightarrow N$ is a smooth map between two Riemannian manifolds (M, g_M) and (N, g_N) , and assume that there exists a constant $c > 0$ such that $f^*g_N < c g_M$. Clearly such a constant exists, if M is compact. Define the rescaled metrics

$$\tilde{g}_M := c g_M, \quad \tilde{g}_N := c^{-1} g_N.$$

One can verify that f is a length (and obviously area) decreasing map with respect to the Riemannian metrics \tilde{g}_M and \tilde{g}_N , as well as with respect to the metrics g_M and g_N . Thus, any smooth map can be made a length decreasing map, if either the domain or the target is scaled appropriately.

- ii) **Totally geodesic maps.** There are plenty of non constant length decreasing minimal maps. For instance, assume that (M, g_M) is a Riemannian manifold and $c \in (0, 1)$ a real constant. The identity map $\text{Id} : (M, g_M) \rightarrow (M, c^{-1}g_M)$ gives a length decreasing minimal map whose graph $\Gamma(\text{Id})$ is even totally geodesic. If σ_M and σ_N are the sectional curvatures of (M, g_M) and $(N, c^{-1}g_M)$, respectively, then

$$\sigma_N = c^{-1}\sigma_M > \sigma_M.$$

Consequently, Theorems C and D are not valid if we assume $\sigma_N > \sigma_M$. Moreover, the assumption $\sigma > 0$ is essential in these theorems and cannot be removed. Indeed, consider the flat 2-dimensional torus $(\mathbb{T}^2, g_{\mathbb{T}})$. By scaling properly the metric $g_{\mathbb{T}}$, the identity map $\text{Id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ produces a length decreasing map. On the other hand, the scaled metric is again flat and Id is certainly neither constant nor an isometry.

Example 3.9. This example shows that there exists an abundance of length decreasing minimal maps that are not totally geodesic.

- i) **Holomorphic maps.** According to the Schwarz-Pick Lemma, any non-linear holomorphic map of the unit disc D in the complex plane \mathbb{C} to itself is strictly length decreasing with respect to the Poincaré metric. The holomorphicity implies that f is a minimal map (cf., [Eel79]). On the other hand, L. Ahlfors [Ahl38] exposed in his generalization of the Schwarz-Pick Lemma the essential role played by the curvature. He proved that if $f : M \rightarrow N$ is a holomorphic map, where N is a Riemann surface with a metric g_N whose Gaussian curvature is bounded from above by a negative constant $-b$ and $M := D$ is the unit disc in \mathbb{C} endowed with an invariant metric g_M whose Gaussian curvature is a negative

constant $-a$, then

$$f^*g_N \leq \frac{a}{b}g_M.$$

Ahlfors' result was extended by S.T. Yau [Yau78] for holomorphic maps between complete Kähler manifolds. More precisely, Yau showed that any holomorphic map $f : M \rightarrow N$, where here M is a complete Kähler manifold with Ricci curvature bounded from below by a negative constant $-a$ and N is a Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant $-b$, then $f^*g_N \leq \frac{a}{b}g_M$.

- ii) **Biholomorphic maps.** Let M be a Kähler manifold and $\text{Aut}(M)$ its *automorphism group*, that is the group of all biholomorphic maps of M . When $m \geq 4$, the group $\text{Aut}(M)$ can be arbitrary large (cf. [Akh95]). This indicates that the results of Theorem D, cannot be extended for the m -Jacobian $\Lambda^m df$. For example, let M be compact, y_0 a fixed point on M , and $f \in \text{Aut}(M)$. Then, the map $\tilde{f} : M \times M \rightarrow M \times M$, $\tilde{f}(x, y) = (f(x), y_0)$, is minimal, as holomorphic, and has identically zero m -Jacobian. In the flat case we can give even explicit examples. For instance, consider the map $f : \mathbb{C}^2 = \mathbb{R}^4 \rightarrow \mathbb{C}^2 = \mathbb{R}^4$, given by

$$f(z, w) := (\beta z + h(w), w), \quad z, w \in \mathbb{C},$$

where $h : \mathbb{C} \rightarrow \mathbb{C}$ is a non-affine holomorphic map and $\beta \leq 1$ a positive real number. Note that the graph $\Gamma(f)$ is minimal in \mathbb{R}^8 , $\|\Lambda^4 df\| = \beta \leq 1$ and f is certainly not an isometry.

Remark 3.10. Let M and N be two Riemannian manifolds satisfying the curvature assumptions in Theorem D. Following essentially the same computations as in the proof of Theorem C, we can prove that the strictly area decreasing property of a map $f : M \rightarrow N$ is preserved under mean curvature flow. The convergence shall be explored in another article where we shall also derive a parabolic analogue of Theorem A.

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